Thm is Let u be a vector space Let x,y,z EV

If x+z = ytz then x=y

Corollary of Thm II Let u be a vector space Let vev

Then (-21) is unique

*PF*

**Suppose**

PE

By VS 6 , 3(-2) EV. By US 4 2 = x + Öv = x+ 2+ (-2) = 9+2+(-2) = y,

suppose that c ev sit. *V+ (-20)'* = *Öv* Then V+ *(-2) =* õv = 2 + (*-*2) By Thm lil (-21)=(-21) There fore (-21) is unique,

Corollary of Thm Il Let v be a vector space Then ou is unique.

PF

Let ue u suppose that *c'*ev sit. ut ã' = *a* Then Utov = v=utori By Thm lil ou or There fore ou is unique,

Thm 1.3

Thm 1,2 Let v be a vector space. Let u EU and a EF (a) Then õo = ou (b) Then (-as) = (-a*)2* = a*(-u)*

PF(a) By VS5, *OU = o*ut ou By VSIO, *02* = (070*)2* = 00 TOU By Thm lil ör= ou

Lett be a vector space. Let W be a subset of u Let wi, wa EW and a EF If õv EW and awit Wa EW Then W is a subspace of V.

ave

PF (b) By US 6, a*u* + *(-40)* = Öv By VSTO, OU + (-a*) 21* = (a-a)2 = 0*01* By Thm 1.2 (a) a*u* +(-a) *a* = on By uniqueness (-a*20)* = (-a) 2*1* By VS, (-a2) = (-9)2 = a-)= *a(-20)* There fore *(-*a*x*) = (-a)2 = a(-21),

PF By Assumption, US 1,2,5 are satisfied Let WEW By thm 1,2 6W) = (-)W ew Thus us 6 also satified. By condition of subspace w is a subspace of V.

Thm 1.8 Let **v be a vector space** Let U E*V.* Then 3! az EF Sit. 0= an po

PF

Thm 1.5 Let u be a vector space. Let W be a subset of v Then span (W) is a subspace of V Move over, any subspace of v that contains W also contain the span(w) **Pf)**

if W = & spanl&s=957 , {o} is subspace of every hector space. if wew, ow = 7 Espan(W) if x.y EW, cxty & span (W) by Condition of subspaces, W is a subspace of every scebspace also vector space, Let sbe a subspace of V. if was then span (W)=So

as BV is generating set, ve span (p"*)* Thug Paz EF st. u= ei ai Bă. suppose that a = žaid" = b:B? Then ū = (3-b.) 85 Stuce B! is Li**nearly Independe**nt, aizbi Thus ai is unique

Thm 1.10 Let U be a vector space. Let G be a generating set of V and IGI = n Let L be a lTnear**ly Independent subset** of V and 141=m

Then

nam

and

PHSG

Sit. LUH generates V and IHC = n-m

Thm 1.9 Let u be a vector space. Let **a be a frunte generating set of V Then maximum linearly indepen*d*ent** subset of G is *ß!* **Hence V has a fratte ba*s*īs.** PF Let W= {91. -.gup be m**aximum Fhearly independent subset of G** Since WSV, span(W) C*V* by thm 1.5 Let g.EG-W Since WU {53 is Linearly indepen*d*ent, 2.9. + 218,7. takgk = o = go = a(a.9.) E span (W) Hence G = spaniw). By thm 1.5 span (G) span (w) = vespan (w) Thus span (W)=V Therefore w is ß

**I**

PF if m=0 L=6 and H=G if m=k suppose that aH if m=ketl L = {(1, ... letih Since (/fler?) WH generates V, Akti is Thear combination of (h/flera?) WH

Thus one of hs Er

is linear Combination

of LUCH-hi?)

Hence L UCH-Chi?) generates v There fore 3H when maktla

Corollary) of Thm 1.10 Let V be a ftrite dimensõon vector space. Then every basīs for u containning the same number of vectors PF

Let IB"l=n and 18V1 = m Since ph is generating set for and *p*lan is līnearly independent set for V By Them 1,10 nam

Corollary 2 of Thm 1:10 **Let v be a vector space with dimensłon n** (a) Let G be a generating set o*f V.*

if Ghave n vectors

Then G = BV (b) Let w **be an ITnearly independent subset of V**

**it w has nuectors**

Then W = BV

(c) Let w **be a maximum line**arly independent set of V

Then W = BV PF *(*a) By Thm 1.9 Subset of G is a basis. and By Corollary L every basī**s have same number of vector.** Thus G should be a basis..

At the same time, e' is also linearly independent set for v and prea is also generating set for V By Thm 1.10 nem

PF (b) By Thm 1.10 $ UW Thus W is basis

is generating set of V. for

There fore

năm,

PF (c)

By Thin WO

TWI SIBI.

always Ip".

Thus maximum lwl is By (b), W=p.

Thm 1.11 Let V be a frutte dim**ensional vector space. Let S be a subspace of v** Then s is finite dimensional & dim (s) = dim (V) More der if dim (s) = dim(V), then u=5 Pf) Since pe is a **linearly in*d*epen*d*ent subset of V** By thim 1.10 1891 3IBU → dim(s) < dim(V) if dim (s) = dim(V), by colleary 2 of Thm 1,10 B = 8 **Therefore** SEV

Thm 2.2 Let u & W be Vector spaces. Let TEL(VW) Then RCT) = span ( Tlf"))

Thm 211 Let U & W be vector spaces Let TEL(V.W) Then NCT) & RCT) are subspaces of v&W.

PF N(T) = { VEV I 7421)=7w Let tlou) = W Then W = Trov) = Tlõutõv)= rou) + TOU) = WtW Thus W = ow an*d ö*v E NCT) Let *U.,0*2 E NCT) and a EF. Then T( 921. + 2x) = a 1*(*21.) + T (20) = ow Thus a2, +2, ENCT) There fore NCT) is subspace of Vo RCTI = { T(U) I DEVI SW Since Tov) = ow, Ow S RCT) Let W., W. ERCT) and aff Then 320., 22 EV Sit. TC2)= W. and TTU.)=W. aw, tw2 = aT(2) + T(02) - Tax,+22) Since az, + EV, *aw*, + We E RCT) Therefore R(T) is subspace of W.

PF Let dim(V)=n Since TCBM) e RCT) and K(T) is subspace of W By Thm 1,5 span ( Treu) ACT) Let r ERCT) Then Bu EU sit. r= T*(2)* ra Tre) - Tra*in*B) e span (T*C*Rs) Thus RCT) & span (TTB')) Therefore RCT) – Spancicev)

Thm 2.3 Lef *V &* W be frite dimensional lector spaces. Let TEL(V.W) Then nulltty (0) + rank (T) = dim()

PF Let B = { ßi,.... on} and BNT) \_f Bi... Bial Let S = BV */ BNCT)* By Thm 2.2 RCT) = span(TIB”) – span (T(s)) suppose that ¿ bi tle%) =ów with some scalars bi@F Since Tis linear rea b*af%) =*őv, *.bißE*NCT*)* Thas 2 6.88 = $c*$ $*c*.p*ht& ba*b*i co As ß" is lhearly independent, bizo Thus tcs) is līncarly independent Consequently, TCS) = BRD because 151 = Itcs), dion (NCT)) + dim(PCT) – dml V).

Thm 2.4 Let V& W be vector spaces Let te *f*(VW) Then I is one-to-one if and only if N(T) = {0} **Pf)** Suppose that T is one-to-one *& 2* E NCT) Then 7(x) = Õw= T(). since I is one to one, we have x=0v

Hence N(T) = {674 Assume that NCD) = *{õre f* 7(2) = T(4), X,YEV Then T (2-y) = T (20) – Tly) = ā Therefore X-Y EN(T) = x-y=0v = x=y This means T is one to ones

**me S**

Thm 2.6 Let V & W be vector spaces. Let dim(V) = n Let Wi..... Wn EW then 3!TEL (V.W) Sit. Tré) = Wi for izl, -,n

Thm 2.5 Let V & W be vector spaces of equal dimensłon Let te f*l*V.W*)* Then it is one to one is equivalent.

T is onto

Trank (T) = dim *(V)* Also we sa*y* T is invertible if and only if rank(t) = dim(V)

PF Let XEV, then a = b antena define T:*V*=W as 7(x) = ça. We Since TIB!) = Wi > Tl Aiß) = QiWin = ATIRO) Thus, Tef(V.W*)* now Suppose that we flv*.W)* and up) = W. Then U(x) = Qi Uff!) = QiWi = T(2) Thus T is unique

*PF*

Suppose that T is one to one. By Thm 2.4, nullity (T) = 0 = By Thm 2:3 rank (T) = dim(u*)* Since V an**d w have equal dimension**, rank(t) = dim(W*) =>* dim (RCT) = dim(W*)* By Thm Will RCTI = W By definition of RIT), RCTI = TCU) Thus TCU) = W T is onto

Corollary of Thm 2.6 Let A E Mmn Then 3!TEL(V.W) St. [1] V = A

PF

Define T by TCe!) = 2, Aic B*u* By Thm 2.63'te 16.W) and it satisfy [1] = A

Thm 2.7 **Let VW be vector spaces over a field F** Let Tu E HIV.W) Then attU : VW is linear and I (VW) = {TIT:V>W & linear } is a **vector space over F.**

Thm 2,8 Let V**&W be finite dimensional vector spaces.** Let B' & BW b**e ordered basis** Let T,U: VW be linear Then [T+07%= [1]+[um and [at] p= atlar

PF

Let x.y EV *&* CEF Then fattu) (cx+y) = at (Octy) + u(ccty)

-CaT620)+atry)+CU0C) + Ucy? = c(ata) u(x)) +at(Y) +Uly)

=c(attu)(x) + (ATHU) CY) Thus attu) is L**inear transformation**

T, E (V.W) & (T+ T.)(x) = T(X) Thus To is og for L (V,W) GT) et(w) sit. (1+ (*-*))(x) = Õw = T.(x) =

PF Let dimn cu) = n & dim (W) = m Then 35A., & a! B.Sot. Tre!) Asi pone and Ures Biblia Then (T+U) (6) = T[e!) +41(8°) - 1 (As: + 8:18"\*

> (ftul): = A: + Bix = (0,3%). Hoewel

» LUI - LT trung and cany B!) = at (8%) = a 2 Anie - (LT)) = a Ace = a1037 → at=aFT

Thm 2,14 **Let v kW be frutte-dimensTonal vector spaces** Let goh kegel **be ordered bases for vew** Let te LV.W)

Then for each nev, [T]p-= [t] (u), PFI) let us & dißt Let W= T(U) - z b; pelo let TCA!) = ? Asi BW Then Thu) - Teža:p!) - $ a: T1B!)

. (I A. (") - 2 (2.. A:) B" – 1 b, .:. by Asindi rbiy rau A12 ... Ain rain

**ANI**

**- M3**

7

frame c

Ibn

Lami Amo ... Amin J L and

Pt2) Let u eV Let fe LCF.V) by fra)=au and geL(F.W) by g(a) = arru) Let BF = {1} be the standard ordered basis for F Then gif

freno - (390) \*\* (as) [14], "-472801 - 67c497x + 27

Thm 2.15 Let A&BE Mmxn (F) Let ou ap is **standard ordered bases fo**r F" \* \* \* Then LA: F" F" t**s Linear** cam [LAIDA (b) LA-LB if and only if A=B *(*) Leate = CLA + LB (1) It T: F">F" is lhear, 3! CE Mmxn Sit. Ta Lc. **In fact** *(C)* If E E Maxp[7] then LAG= LALE (A) If man, then Li= If

PF (d) Let co(7) cm K x EF" By thu 2.14

T(%) = CX = Lc (*3*0) so Tole, C=ETI general 7)- w » [w]ow = [hom minden 5" " 7(2)= W w=[7] 20

al

T*(*X) = w

Pfle) LAE (C:) = AE es = LA (E C) - Laltele:)) By Thm 2.6 LAE = LALE

Let 1.9 € F" af F Then La (ax+y) = A ( ax+y) = a Ax + Ay = a la(w) + Lar*y)* Thus LA is Linear PF*(*a)

Let [Lajona B

By thu 2.4, Lace) = { C. In - Bes they Therefore LLADA

PF (b)

آمار جاء ما فه

[le ]:

By(a*)*, A=B

Thm 2.18. Let V*&*W be vector spaces. let *p*o *& p*u b**e ordered bases .** Let Te Lcv.W) Then T is invertible, if and only if (1) Further more. Crvena CTTO)

**qW**

**is**

**in ve**

Thm 2.17 **Let v*o*w be vector spaces.** Let T be an iso morphism from v to W. Then 74: W → V i**s Thear.** PF Let W.W. EW CEF Since T is onto and one to one, 9.*0.. U*2 E*V* Sit. T(2.) = W. &

*(cw*.fW2) = (CT(2) +T *(*V*.*) = *(* 1*600, +2*.)) *= 64+21*2 \* TCCW, W2) =CW) + (W2),

is invertible

**in VI**

TIV.)=W.

Pf

**Suppose tha**t T is TnvertTble.

Lemma of Thm 2118 Let V *&* W be *v*ector spac*e*s. Let T be an isomorphism **Then V TS finite dimensiona**l if and only if W i**s frutte dimensTonal.** In this case, dim (V) = dim(W) Pf) **Suppose that Vis finite dimensional,** Since T is onto, W = T(V) = RCT) *d*im (W) = dim (RCD)) by Thm 2.2 RCT)= span (Fre')) Hence W 75 fthit**e dimensional by Thm 1,*9*** Since T is one-to-one, nullity (1)=0 by Thm 2,4 By Thm 2.3 range (T) = dim (RCT)) - dim (V) **Therefore dim** (u) = dim CW).

By the Lemma dim(V)= dim (W) = n > Comme Mnen I = [10],= [T" 1] = [t")%E7 Ame where I is identity matrix. Thus (Theme is Tmertible (19) = ([1" how suppose that A- [Theme is invertible. Then 38 € Mnen Sit. AB = BA = I By Thim 2.6 ''UEL(W,V) sit. [1] A - B Then [UT]ev - [u] " (1) " = BA = I - (Islam Thus UT = Iv and similarl*y* TU-IW , Corollary of Thm 2.18 Let V be fini**te dimensional vector space** Let TEL(V) Then T is invertible if and only if [7]is invertible

2.18

Corollary2 of thm Let A E Maxn Then A is Thuertible

if and only if LA i**s invertible.**

Thm 2.19

Thm 2.20 Let v,w be finite **dimensional vector sp**aces and dim (V) en dim(W) =m Then a function $. t(v.W) -> Mmn (F), defined by am =(1) is an isomorphism

Let Viw be finite dimensional vector spaces.

Then

V and W

are

somorphic if and only if

*d*imin) e dim (W)

*PF*

PF **Suppose that V and W are isomorphic** Let T b**e an iso mor phism** from V to W. By Lemina of Thm 2,18, dim (v) = dim (W) **now suppose that di**mcv)= dim(W) By Thm 2:6, 7! T*EL*(.W) sit T(!) – po using Thm 2.2, RCT) = span (Tle')) = span(ple) – W From Thm 2.5 T is also one-to-one. Thus T is an isomorphism from I to w **There fore V and W are isomorphico**

Let TV EL.W) and a EF Then I (att u) = [attujeme = a[i] m + (1 - az(T) + XQ) By Thm 2.8. Thus I is Linear. Let A E Mmm By corollary of Thm 2:6, 3'Tet(V.W) where [] C = A **and every Linear transformation have** matrix hephe senta**tion.** It is shown that is both one to one and onto. Thus į is invertible. **Therefore I** is an isomorphism.,

T is onto

Corollary of Thm 2.20 Let V&W be finite di**mensional vector spaces and di**m cv)=n dim(W)=m Then LV.W) is **finite dimensio**nal & dim *(f(*V*i*w)) = mn

By Thm 220 LV.W) and Mmn (F) are T**somorphic** By Thm 2.19 dim (210,w) = dim[M.mn (F)*]* **Thesetre** dim (L(V.W) = mn.

Thm 2,21 Let u be a finite dimensional Vector space Then per is an isomorphism.

and dim (v) =

Thm 2.22 Let v be a vector space. Let Bh & give be two ordered bases for Let Q = CI] Let VEV Then Q is invertible and

Body = Q Celje

PF

Let 2,0 E*V &* DEF Then , - bier?, Up CH B , autua - 3 (abi tc:)! [au-t 23]qv = a[m.] gx + [2.] eur a p*el 00.4*2.*) – a*gy (en) + f *(*02*)* Thus for is Linear. By thm 1.8 for is both one to one and onto. Thus tv is invertible **Therefore** *a*nd is an isomorphism..

PF Since Iv is By Thom 2.4

invertible, By Thim 2018 Q is Tertible. [0]pra = [Iv (vs] pre = [Julen Frauen = Qu) porn

Thm 2.23 Let v **be a vector space.**

Let evi ell be two ordened bases. bet te LI*V*) Let Q = [Iudeos Then [] eri = Q [T] pra Q and [T] ev k [T] pre are similar.

Thm 3.3 Let V. W be finite dimensional vector spaces. Let ev, pW be ordered bases Let TE L(V.W) Then rankCT) = rank ([T7)

PF)

PF

Let A= cry (= [-] (61- [1,1]) = (17.- [17,0 (1.7.2 [1jxQThen rank city) = rank (La) - rank(t) by PhD 2.4.20 :: [1]pme = @[1]pro Qo

Corollary for Thm 2.23 Let AC Mnen (F) Let Bti be an standard ordered basis for In Let gta be an arbitray ordered basis for Fr Then [ka]gfia = Q*'*AQ

PF

A = [la] pfi & Q= [Julien

Thm 3,4

Let A E Mmn

Let PEMmm , QEMan and both are Tnvertible. Then (a) rank (AQ) = rank(A)

(b) rank(PA) = rankCA) (c) rank CPAQ) = rank(A)

PF la) Relac) = R(Lala) = La La (F") - LA(F") = R(LA) Since LQ is onto. Thus rank (AQ) = rank(A), PF Cb) Since Lp is isomorphism, din (R(LPA)) = dim ( Lp LA (F") - dim ( LA CF”)) - dim *(R*(La)) by prb 2.4.11

Thus rank CPA) = rank(A),

PFCC) dim (R[lpaa)) - dim ( plaka F")) - dîm (La (FM) by Thm 3.4 (6). (b)

Thus rank (PAQ) = rank(A).